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Ordering relations for q-boson operators, continued fraction techniques and the q-CBH enigma

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Abstract. Ordering properties of boson operators have been very extensively studied, and q-analogues of many of the relevant techniques have been derived. These relations have far reaching physical applications and, at the same time, provide a rich and interesting source of combinatorial identities and of their q-analogues. An interesting exception involves the transformation from symmetric to normal ordering, which, for conventional boson operators, can most simply be effected using a special case of the Campbell–Baker–Hausdorff (CBH) formula. To circumvent the lack of a suitable q-analogue of the CBH formula, two alternative procedures are proposed, based on a recurrence relation and on a double continued fraction, respectively. These procedures enrich the repertoire of techniques available in this field. For conventional bosons they result in an expression that coincides with that derived using the CBH formula.

1. Introduction

The boson creation and annihilation operators $a^\dagger$ and a, that satisfy the commutation relation $[a, a^\dagger] \equiv aa^\dagger - a^\dagger a = 1$, give rise to a rich and interesting ordering problem [1]. In the present article we shall for the most part consider operators consisting of monomials with an equal number of creation and annihilation operators. Such operators are diagonal in the number operator basis of the Fock space $\mathcal{F} = \{|k\rangle = \left(1/\sqrt{k!}\right)(a^\dagger)^k|0\rangle;\ k = 0, 1, \cdots\}$. We shall distinguish between normal ordering, for which each monomial is of the form $(a^\dagger)^k a^k$, antinormal ordering, for which the monomials are of the form $a^k(a^\dagger)^k$ and symmetric ordering, for which all operators are expressed in terms of basic elements consisting of averages over all the possible distinct orderings of equal numbers of creation and annihilation operators. In fact, it will be useful to consider the continuous s-ordering, introduced by Cahill and Glauber [2], which reduces to the above three types for $s = 1$, $s = -1$ and $s = 0$, respectively.

Considerable attention has been paid to deformations of the boson commutation relations. The most commonly studied deformed bosons satisfy either the q-commutation relation ('quommutation relation') $[a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = 1$ [3] or the relation $[a, a^\dagger]_q = q^{-\hat{n}}$ [4,5]. $\hat{n}$ is the number operator, that satisfies $[a, \hat{n}]_1 = a$, $[a^\dagger, \hat{n}]_1 = -a^\dagger$. Deformed bosons of the first type have in [6] been nicknamed 'math' bosons, and those of the second type 'phys' bosons.

It is rather obvious that any expression consisting of diagonal monomials can be transformed into an expression depending only on the number operator $\hat{n} = a^\dagger a$, by

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an appropriate number of applications of the commutation relation. One immediate implication of this observation is that $(a^\dagger)^k a^k$ and $a^\ell (a^\dagger)^\ell$, both of which are diagonal in the Fock space, commute with one another. This provides a very transparent inductive proof of the identity $(aa^\dagger a)^m = a^m (a^\dagger)^m a^m$ [7], whose central step is $(aa^\dagger a)^{m+1} = (aa^\dagger a)(a^m (a^\dagger)^m a^m) = a(a^m (a^\dagger)^m)(a^\dagger a)a^m = a^{m+1}(a^\dagger)^{m+1}a^{m+1}$. Turbiner [7] showed, using the actual commutation relations, that this identity holds for a certain subset of deformed bosons (essentially the 'math' bosons). In fact, from the point of view proposed above it is obvious that Turbiner's identity holds for all kinds of deformed boson operators.

The aim of the present study of the interrelations between the various diagonal boson operators is to identify routes towards the extension of these results to q-boson algebras. Thus, while the formulation of a general q-analogue of the Campbell–Baker–Hausdorff (CBH) relation appears not to be straightforwardly feasible [8], an appropriate q-deformed version of the relation between $\exp(\alpha a^\dagger)\exp(\beta a)$ and $\exp(\beta a)\exp(\alpha a^\dagger)$ is used to generate the transformation between normal and antinormal orderings. A principal problem involves the generalization of the CBH relation between $\exp(\alpha a^\dagger + \beta a)$ and $\exp(\alpha a^\dagger)\exp(\beta a)$, that is the most direct means to derive the transformation from symmetric to normal ordering. In the absence of a corresponding q-deformed CBH formula, we develop two alternative procedures, involving recurrence relations and double continued fractions, respectively. These procedures provide a transformation from a symmetrically ordered expression into an expression in terms of the number operator. The normally ordered expansion can easily be derived from the latter form. For conventional bosons these procedures result in expressions that coincide with those obtained using the appropriate CBH formula, but unlike the latter they can easily be extended to the treatment of deformed bosons. The q-deformation of the CBH relation has also been considered, from very different points of view, in [9–11].

This article is organized as follows. In section 2 we review the simplest type of transformations between differently ordered expressions, using straightforward recurrence relation techniques, and introduce a (p, q)-boson algebra automorphism that accounts for some symmetries between different ordering relations. Elementary combinatorial identities that play a useful role in the derivation of reordering transformations are introduced where appropriate. We consider the 'math'-type q-deformed case, from which the conventional case can be easily obtained. In section 3 we present the CBH-based normal to antinormal and symmetric to normal transformations, and the q-analogue of the former, that uses a q-CBH type identity derived by McDermott and Solomon [12]. A recurrence relation procedure for the transformation from symmetric ordering to a number-operator expansion, that is applicable to both undeformed and deformed bosons, is presented in section 4, and a corresponding double continued fraction approach is introduced in section 5. Some concluding remarks, indicating desirable further effort, are made in section 6.

2. Ordering transformations via elementary recurrence relations and combinatorial identities

In this section we consider the most direct and elementary approaches to the boson operator ordering problem. Recurrence relations between consecutive transformation coefficients of various types are derived and used to obtain closed form expressions for the coefficients, as far as possible. Whenever possible, we present the q-boson result, from which the $q \to 1$ limit can readily be obtained.

Some immediate consequences of the 'math' type q-boson quommutation relation

$[a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = 1$ are

$$[a, (a^\dagger)^k]_{q^k} = [k]_q (a^\dagger)^{k-1} \qquad [a^k, a^\dagger]_{q^k} = [k]_q a^{k-1} \tag{1}$$

where

$$[k]_q = \frac{q^k - 1}{q - 1}.$$

The deformed Fock space is spanned by

$$\left\{ |k\rangle = \frac{1}{\sqrt{[k]_q!}} (a^\dagger)^k |0\rangle; \quad k = 0,\ 1,\ 2,\ \ldots \right\}$$

where $[k]_q! \equiv [k-1]_q![k]_q$ and $[0]_q! \equiv 1$. It follows that $\hat{n}|k\rangle = k|k\rangle$, $a^\dagger|k\rangle = \sqrt{[k+1]_q}|k+1\rangle$, $a|k\rangle = \sqrt{[k]_q}|k-1\rangle$ and $a^\dagger a|k\rangle = [k]_q|k\rangle$. Consequently, on the Fock space $a^\dagger a = [\hat{n}]_q$ and $aa^\dagger = [\hat{n}+1]_q$. Since on the Fock space a and $a^\dagger$ are Hermitian conjugates of one another, the second of equations (1) is the Hermitian conjugate of the first.

Some q-arithmetic relations that will be used below are

$$[i+j]_q = [i]_q + q^i [j]_q$$

$$[i-j]_q = q^{-j}([i]_q - [j]_q) = [i]_q - q^{i-j}[j]_q$$

$$[i+j]_q[i-j]_q = [i]_q^2 - q^{i-j}[j]_q^2.$$

We now proceed to consider various types of transformations that are amenable to a straightforward recurrence relation formulation.

2.1. q-number to q-normal ordering

The recurrence relation for the transformation coefficients in the relation

$$[\hat{n}]_q^k = \sum_{\ell=1}^{k} \tilde{C}_{k,\ell} (a^\dagger)^\ell a^\ell \tag{2}$$

i.e.

$$\tilde{C}_{k+1,\ell} = q^{\ell-1}\tilde{C}_{k,\ell-1} + [\ell]_q \tilde{C}_{k,\ell}$$

was derived in [13], where the coefficients were recognized as the q-Stirling numbers of the second kind. The latter were originally introduced by Carlitz[14] and studied by Gould [15]. Some recent developments and generalizations are presented in [16].

The transformation from q-number to q-normal ordering can also be obtained by noting that

$$(a^\dagger)^k a^k |n\rangle = [n]_q [n-1]_q \cdots [n-k+1]_q |n\rangle \tag{3}$$

to obtain the operator identity

$$(a^\dagger)^k a^k = [\hat{n}]_q [\hat{n}-1]_q \cdots [\hat{n}-k+1]_q = \frac{(1-(1-q)[\hat{n}]_q; q^{-1})_k}{(1-q)^k} \tag{4}$$

where $(a; q)_n = \prod_{i=1}^{n}(1 - aq^{i-1})$ [17]. Recalling the defining relation for the q-Stirling numbers of the second kind

$$[x]_q^k = \sum_{\ell=1}^{k} S_q(k,\ell)[x]_q[x-1]_q \cdots [x-\ell+1]_q$$

equation (2) follows.

2.2. q-number to q-antinormal ordering

Writing $[\hat{n}]_q^k = \sum_{\ell=0}^{k} \tilde{d}_{k,\ell} a^\ell (a^\dagger)^\ell$ one obtains the recurrence relation

$$\tilde{d}_{k+1,\ell} = \frac{1}{q^\ell} \tilde{d}_{k,\ell-1} - \frac{1}{q^{\ell+1}} [\ell+1]_q \tilde{d}_{k,\ell} \,. \tag{5}$$

Noting that $\tilde{d}_{0,0} = 1$, the transformation coefficients can be systematically evaluated.

2.3. q-antinumber to q-normal ordering

Defining the antinumber operator $\check{n}$ through the relation $[\check{n}]_q = aa^\dagger = qa^\dagger a + 1$ we write $[\check{n}]_q^k = \sum_{\ell=0}^{k} \tilde{d}'_{k,\ell} (a^\dagger)^\ell a^\ell$ and note that $\tilde{d}'_{0,0} = 1$. The following recurrence relation is obtained:

$$\tilde{d}'_{k+1,\ell} = q^\ell \tilde{d}'_{k,\ell-1} + [\ell+1]_q \tilde{d}'_{k,\ell} \,. \tag{6}$$

From the defining relation it follows that $\check{n} = \hat{n} + 1$, so that the antinumber operator, that could more simply be referred to as the shifted number operator, does not appear to deserve separate attention. However, comparing equations (5) and (6) we note that

$$\tilde{d}_{k,\ell}(q) = \left(-\frac{1}{q}\right)^{k-\ell} \tilde{d}'_{k,\ell}\left(\frac{1}{q}\right) .$$

The origin of this symmetry property, and similar symmetries encountered below, is accounted for in the following subsection.

2.4. A boson operator algebra automorphism and its consequences

In the present subsection we introduce a transformation between the creation and the annihilation operators that will be found useful subsequently.

Rewriting the quommutation relation in the form $paa^\dagger - qa^\dagger a = 1$, the (p,q)-number to (p,q)-antinormal and the (p,q)-antinumber to (p,q)-normal recurrence relations become

$$\tilde{d}_{k+1,\ell} = \left(\frac{p}{q}\right)^\ell \tilde{d}_{k,\ell-1} - \frac{1}{q^{\ell+1}} [\ell+1]_{p,q} \tilde{d}_{k,\ell}$$

and

$$\tilde{d}'_{k+1,\ell} = \left(\frac{q}{p}\right)^\ell \tilde{d}'_{k,\ell-1} + \frac{1}{p^{\ell+1}} [\ell+1]_{p,q} \tilde{d}'_{k,\ell}$$

where

$$[k]_{p,q} = \frac{p^k - q^k}{p-q} .$$

Under the algebra automorphism

$$\psi : \begin{cases} a & \to & -a^\dagger \\ a^\dagger & \to & a \\ q & \to & p \\ p & \to & q \end{cases} \tag{7}$$

the modified quommutation relation remains invariant. Under this isomorphism $\hat{n} \to -\check{n}$ and $a^\ell (a^\dagger)^\ell \to (-1)^\ell (a^\dagger)^\ell a^{ell}$. Applying these transformations to the (p,q)-number to (p,q)-antinormal or to the (p,q)-antinumber to (p,q)-normal expansions we note that $\tilde{d}_{k,\ell}(q,p) = (-1)^{k-\ell} \tilde{d}'_{k,\ell}(p,q)$.

2.5. q-normal to q-number

Writing

$$(a^\dagger)^k a^k = \sum_{\ell=1}^{k} \tilde{c}_{k,\ell} [\hat{n}]_q^\ell \tag{8}$$

we obtain

$$(a^\dagger)^{k+1} a^{k+1} = a^\dagger \Big((a^\dagger)^k a^k \Big) a = a^\dagger \left(\sum_{\ell=1}^{k} \tilde{c}_{k,\ell} [\hat{n}]_q^\ell \right) a = \sum_{\ell=1}^{k} \tilde{c}_{k,\ell} [\hat{n} - 1]_q^\ell \hat{n}$$

$$= \sum_{\ell=1}^{k} \tilde{c}_{k,\ell} \frac{1}{q^\ell} ([\hat{n}]_q - 1)^\ell [\hat{n}]_q$$

from which the recurrence relation

$$\tilde{c}_{k+1,\ell} = \sum_{j=\ell-1}^{k} \tilde{c}_{k,j} q^{-j} \binom{j}{\ell - 1} (-1)^{j-\ell-1} \tag{9}$$

results. This is the 'wrong' way to obtain a recurrence relation for the coefficients.

The 'right' way

$$(a^\dagger)^{k+1} a^{k+1} = (a^\dagger)^k [\hat{n}]_q a^k = [\hat{n} - k]_q \sum_{\ell=1}^{k} \tilde{c}_{k,\ell} [\hat{n}]_q^\ell = \sum_{\ell=1}^{k+1} (\tilde{c}_{k,\ell-1} - [k]_q \tilde{c}_{k,\ell}) q^{-k} [\hat{n}^\ell]_q$$

was presented in [13], yielding the recurrence relation

$$\tilde{c}_{k+1,\ell} = (\tilde{c}_{k,\ell-1} - [k]_q \tilde{c}_{k,\ell}) q^{-k} \tag{10}$$

that, along with the initial value $\tilde{c}_{1,1} = 1$, identifies the coefficients $\tilde{c}_{k,\ell}$ as the q-Stirling numbers of the first kind, $s_q(k, \ell)$. Equation (9) is a dual recurrence relation for these q-Stirling numbers.

Equivalently, using equation (3) along with the defining relation for the q-Stirling numbers of the first kind

$$[x]_q [x-1]_q \cdots [x - k + 1]_q = \sum_{\ell=0}^{k} s_q(k, \ell) [x]_q^\ell$$

we obtain equation (8).

2.6. q-normal to q-antinormal ordering

Write

$$(a^\dagger)^k a^k = \sum_{\ell=0}^{k} \tilde{h}_{k,\ell} a^\ell (a^\dagger)^\ell \tag{11}$$

Obviously, $\tilde{h}_{0,0} = 1$.

Noting that $(a^\dagger)^{k+1} a^{k+1} = (a^\dagger)^k [\hat{n}]_q a^k = (a^\dagger)^k a^k [\hat{n} - k]_q$ we obtain

$$(a^\dagger)^{k+1} a^{k+1} = \sum_{\ell=0}^{k} \tilde{h}_{k,\ell} a^\ell (a^\dagger)^\ell [\hat{n} - k]_q$$

$$= \sum_{\ell=0}^{k} \tilde{h}_{k,\ell} a^\ell [\hat{n} - (k + \ell)]_q (a^\dagger)^\ell$$

$$= \sum_{\ell=0}^{k} \tilde{h}_{k,\ell} a^{\ell} q^{-(k+\ell+1)} (aa^{\dagger} - [k+\ell+1]_q)(a^{\dagger})^{\ell} =$$

$$= \sum_{\ell=1}^{k+1} \tilde{h}_{k,\ell-1} q^{-(k+\ell)} a^{\ell} (a^{\dagger})^{\ell} - \sum_{\ell=0}^{k} \tilde{h}_{k,\ell} q^{-(k+\ell+1)} [k+\ell+1]_q a^{\ell} (a^{\dagger})^{\ell}$$

Hence

$$\tilde{h}_{k+1,\ell} = q^{-(k+\ell)} \tilde{h}_{k,\ell-1} - q^{-(k+\ell+1)} [k+\ell+1]_q \tilde{h}_{k,\ell} . \tag{12}$$

This recurrence relation, as well as the initial condition, are satisfied by

$$\tilde{h}_{k,\ell} = (-1)^{k-\ell} [k-\ell]_q! \begin{bmatrix} k \\ \ell \end{bmatrix}_q^2 q^{-\frac{1}{2}(k-\ell)(k+\ell+1)-k\ell} . \tag{13}$$

The inverse, q-antinormal to q-normal ordering,

$$a^k (a^{\dagger})^k = \sum_{\ell=0}^{k} \tilde{H}_{k,\ell} (a^{\dagger})^{\ell} a^{\ell} \tag{14}$$

can most readily be obtained by invoking the algebra automorphism discussed in subsection 2.4. The latter implies that

$$\tilde{H}_{k,\ell}(q) = \left(-\frac{1}{q}\right)^{k-\ell} \tilde{h}_{k,\ell}\left(\frac{1}{q}\right) = q^{\ell^2} [k-\ell]_q! \begin{bmatrix} k \\ \ell \end{bmatrix}_q^2 \tag{15}$$

a relation that is easily verified directly.

Noting that

$$a^k (a^{\dagger})^k |n\rangle = [n+1]_q [n+2]_q \cdots [n+k]_q |n\rangle$$

we obtain

$$a^k (a^{\dagger})^k = [\hat{n}+1]_q [\hat{n}+2]_q \cdots [\hat{n}+k]_q = \frac{(1-(1-q)[\hat{n}+1]_q; q)_k}{(1-q)^k} . \tag{16}$$

Applying the q-Chu–Vandermonde sum (equation II.6 in [17]) in two different ways we obtain both (11) (with the coefficients agreeing with (13)) and (14) (with the coefficients agreeing with (15)).

2.7. *Consecutive transformations*

It is of some interest to inspect some of the consistency relations obtained by considering the commutativity of diagrams of the form

$$\begin{tikzcd} \mathrm{A} \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & \mathrm{B} \arrow[d, "\beta"] \\ & \mathrm{C} \end{tikzcd}$$

where A, B and C stand for diagonal expressions in different forms and α, β and γ stand for the appropriate transformations, i.e. $\alpha : a \mapsto b$, $\beta : b \mapsto c$ and $\gamma : a \mapsto c$, where $a \in A$, $b \in B$ and $c \in C$. Equivalently, $a\alpha = b$, $b\beta = c$ and $a\gamma = c$, i.e. $\gamma = \alpha \circ \beta$.

As an example we consider the following.

Normal to number to antinormal ordering.

$$(a^\dagger)^k a^k = \sum_{\ell}^{k} C_{k,\ell}\hat{n}^k = \sum_{\ell=0}^{k} C_{k,\ell} \sum_{m=0}^{\ell} d_{\ell,m} a^m (a^\dagger)^m = \sum_{\ell=0}^{k} a^\ell (a^\dagger)^\ell \sum_{m=\ell}^{k} C_{k,m} d_{m,\ell}$$

i.e.

$$h_{k,\ell} = \sum_{m=\ell}^{k} C_{k,m} d_{m,\ell} \,.$$

Other transformations of this kind appear subsequently.

3. Exponential generating functions and the Cahill–Glauber s-ordering

Cahill and Glauber [2] pointed out that

$$\begin{aligned} D(\alpha) &\equiv \exp(\alpha a^\dagger - \alpha^* a) \\ &= \exp\left(-\frac{|\alpha|^2}{2}\right)\exp(\alpha a^\dagger)\exp(-\alpha^* a) \\ &= \exp\left(\frac{|\alpha|^2}{2}\right)\exp(-\alpha^* a)\exp(\alpha a^\dagger) \end{aligned}$$

generates the symmetric, normal and antinormal orderings, respectively:

$$\begin{aligned} D(\alpha) &= \sum_{n,m=0}^{\infty} \frac{\alpha^n}{n!}\frac{(-\alpha^*)^m}{m!}\{(a^\dagger)^n a^m\} \\ &= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n,m=0}^{\infty}\frac{(\alpha a^\dagger)^n}{n!}\frac{(-\alpha^* a)^m}{m!} \\ &= \exp\left(\frac{|\alpha|^2}{2}\right)\sum_{n,m=0}^{\infty}\frac{(-\alpha^* a)^m}{m!}\frac{(\alpha a^\dagger)^n}{n!} \,. \end{aligned}$$

The symbol $\{(a^\dagger)^n a^m\}$ is the sum of the $\binom{n+m}{n}$ monomials consisting of n factors $a^\dagger$ and m factors a, divided by $\binom{n+m}{n}$.

They introduced the s-ordered displacement operator $D(\alpha, s) = D(\alpha)\exp(\frac{1}{2}s|\alpha|^2)$ in terms of which they defined the s-ordered product $\{(a^\dagger)^n a^m\}_s$ via the expansion

$$D(\alpha, s) = \sum_{n,m=0}^{\infty} \frac{\alpha^n(-\alpha^*)^m}{n!m!}\{(a^\dagger)^n a^m\}_s \,.$$

The s-ordered product reduces to the normal, symmetric and antinormal orderings for $s = 1$, 0 and -1, respectively.

For the diagonal s-ordered term $\{(a^\dagger)^k a^k\}_s$ one obtains

$$\{(a^\dagger)^k a^k\}_s = \sum_{j=0}^{k} \binom{k}{j}^2 (k-j)!\left(\frac{1-s}{2}\right)^{k-j}(a^\dagger)^j a^j$$

which, for the normally ordered case ($s = 1$) reduces to the obvious relation $\{(a^\dagger)^k a^k\}_1 = (a^\dagger)^k a^k$, and for the antinormally ordered expression yields

$$\{(a^\dagger)^k a^k\}_{-1} \equiv a^k (a^\dagger)^k = \sum_{\ell=0}^{k}\binom{k}{\ell}^2 (k-\ell)!(a^\dagger)^\ell a^\ell \,.$$

Using the transformation from normal to number ordering

$$(a^\dagger)^k a^k = \sum_{\ell=0}^{k} s(k,\ell)\hat{n}^\ell$$

we obtain

$$\{(a^\dagger)^k a^k\}_s = \sum_{\ell=0}^{k} \hat{n}^\ell \alpha_s(k,\ell)$$

where

$$\alpha_s(k,\ell) = \sum_{i=0}^{k-\ell} \binom{k}{i}^2 i! \left(\frac{1-s}{2}\right)^i s(k-i,\ell)$$

and where the definition of the Stirling numbers of the first kind has been extended to include $s(\ell,0) \equiv \delta_{\ell,0}$. The latter is consistent with the definition

$$\ell!\binom{x}{\ell} = \sum_{m=1}^{\ell} s(\ell,m)x^m = \sum_{m=0}^{\ell} s(\ell,m)x^m$$

where the last equality implies $s(\ell,0) = \delta_{\ell,0}$.

We now proceed to consider the q-analogues of some of the Cahill–Glauber transformations.

3.1. *q-normal to q-antinormal ordering*

The following q-deformed analogue of one special case of the CBH relation has been derived by McDermott and Solomon [12]

$$\exp_q(\alpha a^\dagger)\exp_q(\beta a) = \exp_q(\beta a)\exp_q(-\beta\alpha)\exp_q(\alpha a^\dagger) \tag{17}$$

where $[\alpha,\beta]_q = 0$ and where α and β commute with a and $a^\dagger$. $\exp_q(x) \equiv \sum_{i=0}^{\infty} x^i/[i]_q!$ is Jackson's q-exponential [18]. In the notation used by Gasper and Rahman [17] $\exp_q(z) = e_q((1-q)z)$. Note that $[\beta a, \alpha a^\dagger] = \beta\alpha$, $[\alpha a^\dagger, \beta\alpha]_q = 0$ and $[\beta\alpha, \beta a]_q = 0$.

A minor generalization of (17), i.e.

$$\exp_q(A)\exp_q(B) = \exp_q(B)\exp_q(-[B,A]_1)\exp_q(A) \tag{18}$$

where $[[A,B]_1,B]_q \equiv [[A,B]_q,B]_1 = 0$ and $[A,[B,A]_1]_q \equiv [[A,B]_q,A]_1 = 0$, was derived by Kashaev [19], citing a private communication by A Yu Volkov. The special case $[A,B]_q = 0$ is due to Faddeev and Kashaev [20] and was presented and used by Kirillov [21]. For $q = 1$ equation (18) reduces to $\exp(A)\exp(B) = \exp(B)\exp(-[B,A])\exp(A)$. The last identity coincides with a familiar special case of the CBH formula, obtained when $[B,A]$ commutes with both A and B.

Expanding both sides of (17) in a double power series in α and β (paying attention to the fact that these two quantities do not commute) and using the identities

$$\alpha^k\beta^\ell = q^{k\ell}\beta^\ell\alpha^k$$

and

$$(\beta\alpha)^k = q^{-k(k+1)/2}\alpha^k\beta^k$$

we obtain

$$(a^\dagger)^i a^j = \sum_{\ell=0}^{\min(i,j)} (-1)^\ell [\ell]_q! \begin{bmatrix} i \\ \ell \end{bmatrix}_q \begin{bmatrix} j \\ \ell \end{bmatrix}_q q^{\frac{1}{2}\ell(\ell-1)-ij} a^{j-\ell}(a^\dagger)^{i-\ell}. \tag{19}$$

For $i = j$ this reduces to (11), with the coefficient given by (13).

3.2. *q-antinormal to q-normal ordering*

Multiplying equation (17) by $\exp_{q^{-1}}(\beta a)$ from both the right and the left and recalling that $\exp_q(-x)\exp_{q^{-1}}(x) = 1$, cf [18], we obtain

$$\exp_{q^{-1}}(\beta a)\exp_{q^{-1}}(\alpha a^\dagger) = \exp_{q^{-1}}(\alpha a^\dagger)\exp_{q^{-1}}(\beta\alpha)\exp_{q^{-1}}(\beta a).$$

Expanding in powers of α and β as in the previous subsection, we obtain

$$a^i(a^\dagger)^j = \sum_{\ell=0}^{\min(i,j)} q^{\ell(\ell-i-j)+ij}[\ell]_q! \begin{bmatrix} i \\ \ell \end{bmatrix}_q \begin{bmatrix} j \\ \ell \end{bmatrix}_q (a^\dagger)^{j-\ell}a^{i-\ell}. \tag{20}$$

This result agrees with equation (3.1.2) of [22], and, for $i = j$, reduces to (14) with the coefficient given by (15).

3.3. *The elusive q-analogue of the symmetric to normal ordering transformation*

A recent study [8] pointed out that a q-analogue of the general CBH formula is not straightforwardly derivable. This does not preclude more subtle generalizations or generalizations of specific types of special cases, such as the relation used in the preceding two subsections. However, we have not been able to derive a q-analogue of the relation between $\exp(\alpha a^\dagger + \beta a)$ and $\exp(\alpha a^\dagger)\exp(\beta a)$, that one would need to effect the symmetric to q-normal ordering transformation. The latter transformation will therefore be considered in the following two sections, introducing techniques that one may view as substitutes for this elusive q-CBH formula.

4. A recurrence relation for symmetric to number operator ordering

The action of the diagonal operators $\hat{n}^k$, $(a^\dagger)^k a^k$ and $a^k(a^\dagger)^k$ in the Fock space can be written down immediately, as was done in the appropriate places above. However, this is not the case for the symmetrically ordered operator $\{(a^\dagger)^k a^k\}$.

Starting from $|n\rangle$ we note that $\{a^\dagger a\}|n\rangle = \frac{1}{2}(a^\dagger a|n\rangle + aa^\dagger|n\rangle)$, i.e. the eigenvalue of $\{a^\dagger a\}$ is obtained by averaging over the two two-step routes from $|n\rangle$ back to $|n\rangle$ generated by $a^\dagger a$ and $aa^\dagger$, and weighted by $\sqrt{[n+1]_q}\sqrt{[n+1]_q} = [n+1]_q$ and $\sqrt{[n]_q}\sqrt{[n]_q} = [n]_q$, respectively.

For $\{(a^\dagger)^2a^2\}$ we have to allow the six four-step diagonal routes $(a^\dagger)^2a^2$, $a^\dagger aa^\dagger a$, $a^\dagger a^2a^\dagger$, $a(a^\dagger)^2a$, $aa^\dagger aa^\dagger$ and $a^2(a^\dagger)^2$. Note that the steps are weighted horizontally, i.e. a step connecting $|n\rangle$ and $|n+1\rangle$ in either direction is weighted by $\sqrt{[n+1]_q}$

$$W_{m'\to m} = \sqrt{[\max(m,m')]_q}\,\delta_{|m-m'|,1}.$$

This can also be expressed as $W_{n\rightleftharpoons n+1} = \sqrt{[n+1]_q}$, all other weights vanishing. The weight of a multi-step route is obtained by multiplying the weights of the steps of which the route consists. Therefore, the sum of the weights of the k-step routes from $|n\rangle$ to $|m\rangle$ can be obtained recursively, summing over all the $k-1$ step routes from which $|m\rangle$ can be reached with one more step

$$W^{(k)}_{n\to m} = \sum_{m'} W^{(k-1)}_{n\to m'}W_{m'\to m} = W^{(k-1)}_{n\to m-1}\sqrt{[m]_q} + W^{(k-1)}_{n\to m+1}\sqrt{[m+1]_q}$$

Note that the steps are weighted in such a way that if a route involves a node with $m < 0$ its weight vanishes automatically, so that no explicit precautions need be taken to avoid such routes.

Extracting the irrational factor

$$\left(\frac{[\max(m,n)]_q!}{[\min(m,n)]_q!}\right)^{\frac{1}{2}}$$

in $W^{(k)}_{n\to m}$ we define the reduced (rational) weights $\omega^{(k)}_{n\to m}$ via the relation

$$W^{(k)}_{n\to m} = \left(\frac{[\max(m,n)]_q!}{[\min(m,n)]_q!}\right)^{\frac{1}{2}} \omega^{(k)}_{n\to m}\,.$$

The reduced weights satisfy the recurrence relation

$$\omega^{(k)}_{n\to m} = \omega^{(k-1)}_{n\to m-1}\phi_{n,m}(m) + \omega^{(k-1)}_{n\to m+1}\phi_{m,n}(m+1)$$

where

$$\phi_{a,b}(c) \equiv \begin{cases} [c]_q & \text{if } b \leqslant a \\ 1 & \text{otherwise.} \end{cases}$$

It is now obvious that

$$W^{(k+\ell)}_{n\to n+\ell-k} = \binom{k+\ell}{\ell} < n+\ell-k|\{(a^\dagger)^\ell a^k\}|n\rangle = \sqrt{\frac{\max(n+\ell-k,n)!}{\min(n+\ell-k,n)!}} \sum_{i=0}^{\min(k,\ell)} \beta_i^{(k,\ell)} n^i\,. \tag{21}$$

Moreover, the leading coefficient is $\beta^{(k,\ell)}_{\min(k,\ell)} = \binom{k+\ell}{\ell}$, the number of $(k+\ell)$-routes from $|n\rangle$ to $|n+k-\ell\rangle$, i.e. the number of terms in $\{(a^\dagger)^\ell a^k\}$.

Equation (21) can be written as the operator identity

$$\{(a^\dagger)^\ell a^k\} = \binom{k+\ell}{\ell}^{-1} \begin{cases} (a^\dagger)^{\ell-k}\sqrt{[\hat n+1]_q[\hat n+2]_q\cdots[\hat n+\ell-k]_q}\,\omega^{(2k)}_{\hat n\to\hat n+\ell-k} & \ell \geqslant k \\ a^{k-\ell}\sqrt{[\hat n]_q[\hat n-1]_q\cdots[\hat n+\ell-k+1]_q}\,\omega^{(2k)}_{\hat n\to\hat n+\ell-k} & \ell < k\,. \end{cases}$$

To proceed systematically it is convenient to consider the following three distinct cases

$$\omega^{(k)}_{n\to n+k-2\ell} = \omega^{(k-1)}_{n\to n+k-2\ell-1} + \omega^{(k-1)}_{n\to n+k-2\ell+1}[n+k-2\ell+1]_q \qquad 0 \leqslant \ell < \left\lfloor\frac{k}{2}\right\rfloor$$

$$\omega^{(k)}_{n\to n} = \omega^{(k-1)}_{n\to n-1}[n]_q + \omega^{(k-1)}_{n\to n+1}[n+1]_q$$

$$\omega^{(k)}_{n\to n-k+2\ell} = \omega^{(k-1)}_{n\to n-k+2\ell-1}[n-k+2\ell]_q + \omega^{(k-1)}_{n\to n-k+2\ell+1} \qquad 0 \leqslant \ell < \left\lfloor\frac{k}{2}\right\rfloor.$$

For $\ell = 0$ we start from $\omega^{(0)}_{n\to n} = 1$ and note that $\omega^{(k)}_{n\pm(k+1)} = 0$, to obtain

$$\omega^{(k)}_{n\to n\pm k} = 1\,.$$

Using this result we obtain for $\ell = 1$ and $k > 2$ the two-term recurrence relation

$$\omega^{(k)}_{n\to n+k-2} = \omega^{(k-1)}_{n\to n+k-3} + [n+k-1]_q$$

from which, along with

$$\omega^{(2)}_{n\to n} = [n]_q + [n+1]_q \tag{22}$$

we obtain

$$\omega^{(k)}_{n\to n+k-2} = [k]_q[n]_q + \frac{[k]_q - k}{q-1}$$

that for $q = 1$ reduces to

$$\omega^{(k)}_{n\to n+k-2} = kn + \frac{k(k-1)}{2}\,. \tag{23}$$

Similarly

$$\omega^{(k)}_{n\to n-k+2} = [n+2-k]_q[k]_q + \frac{[k]_q - k}{q-1}$$

or, in the $q = 1$ limit

$$\omega^{(k)}_{n\to n-k+2} = kn - \frac{k(k-3)}{2}\,. \tag{24}$$

Obtaining

$$\omega^{(4)}_{n\to n} = [3]_q[n]_q\Big([n-1]_q + [n+1]_q\Big) + \frac{[3]_q - 3}{q-1}\Big([n]_q + [n+1]_q\Big)$$

we proceed to write the two-term recurrence relation

$$\omega^{(k)}_{n\to n+k-4} = \omega^{(k-1)}_{n\to n+(k-1)-4} + \left([k-1]_q[n]_q + \frac{[k-1]_q - (k-1)}{q-1}\right)[n+k-3]_q\,.$$

In the limit $q = 1$ we have

$$\omega^{(4)}_{n\to n} = 6(n^2 + n + \tfrac{1}{2}) \tag{25}$$

and the recurrence relation yields

$$\omega^{(k)}_{n\to n+k-4} = \binom{k}{k-2}\left(n^2 + (k-3)n + \frac{(k-3)(k-2)}{4}\right)\,. \tag{26}$$

Similarly

$$\omega^{(k)}_{n\to n-k+4} = \binom{k}{k-2}\left(n^2 - (k-5)n + \frac{(k-5)(k-4)}{4} + \frac{1}{2}\right)\,. \tag{27}$$

etc.

4.1. Symmetric ordering to number operator expansion

Comparison with (21) suggests that

$$\{(a^\dagger)^k a^k\} = \binom{2k}{k}^{-1} \omega^{(2k)}_{n\to n}\Big|_{n=\hat{n}}\,.$$

Thus, from (23) or (24) we obtain $\{a^\dagger a\} = \hat{n} + \frac{1}{2}$ and from (26) or (27) $\{(a^\dagger)^2 a^2\} = \hat{n}^2 + \hat{n} + \frac{1}{2}$. More generally

$$\{(a^\dagger)^\ell a^k\} = \binom{k+\ell}{\ell}^{-1} A^{(\ell-k)} \omega^{(k+\ell)}_{n\to n+\ell-k}\Big|_{n=\hat{n}} \equiv \binom{k+\ell}{\ell}^{-1} A^{(\ell-k)} f_{\min(k,\ell)}(\hat{n})$$

where

$$A^{(\ell-k)} = \begin{cases} (a^\dagger)^{\ell-k} & \ell \geqslant k \\ a^{k-\ell} & \ell < k \end{cases}$$

and $f_m(\hat{n})$ is a polynomial in $\hat{n}$ of degree m.

This expression can be written in the form

$$\{(a^\dagger)^\ell a^k\} = \binom{k+\ell}{\ell}^{-1} \begin{cases} (a^\dagger)^{\ell-k} f_k(\hat{n}) & \text{if } \ell \geqslant k \\ f_\ell(\hat{n}+k-\ell)a^{k-\ell} & \text{if } \ell < k\,. \end{cases}$$

Along with the transformation from number to normal ordering it provides a relation of the form

$$\{(a^\dagger)^\ell a^k\} = \sum \eta_i^{(k,\ell)} (a^\dagger)^{\ell-k+i} a^i\,.$$

where the coefficients are easily obtained by summing over products of coefficients corresponding to the two constituent transformations.

5. Continued fraction representation of the generating series for the reduced weights

In this section we compute the generating series

$$T_{n\to n+k} := \sum_{i\geqslant 0} t^i \omega^{(i)}_{n\to n+k} \tag{28}$$

defined for $k \geqslant -n$.

We use the tool of non-commutative series which leads to a continued fraction expression of (28). The results are stated in the context of the q-Weyl algebra.

Let V be a vector space over a field k with basis $|e_n\rangle$ $n = 0, 1, \ldots$ and equipped with its natural grading

$$V = \oplus_{n\in\mathbb{Z}} V_n \quad \text{with } V_n := ke_n \qquad V_{-n-1} := (0) \quad \text{for } n \geqslant 0$$

and $f,\ g$ two linear operators in V of degrees $-1,\ +1$, respectively. That is

$$\text{for } n \geqslant 0 \qquad f|e_0\rangle := 0 \qquad f|e_{k+1}\rangle = \alpha_{k+1}|e_k\rangle \qquad g|e_k\rangle := \beta_k|e_{k+1}\rangle\,.$$

Consider the words in $f,\ g$:

$$w(f,g) := f^{p_1} g^{q_1} f^{p_2} g^{q_2} \cdots f^{p_n} g^{q_n}$$

the degree of such an operator is $\sum_{k=1}^n (q_k - p_k)$. Obviously, V is a generalized Fock space.

We recall here the basic properties of non-commutative series and, for details, we refer the reader to [23] or [24].

Let $\mathcal{A} := \{u, v\}$ be a two letter alphabet. The set of finite sequences (called *words*)

$$x_1 x_2 \cdots x_n \qquad x_i \in \mathcal{A} \quad n \in \mathbb{N}$$

is the *free monoid* generated by $\mathcal{A}$ and denoted $\mathcal{A}^*$. The product law of this monoid is the *concatenation* of sequences

$$x_1 x_2 \cdots x_n . y_1 y_2 \cdots y_m := x_1 x_2 \cdots x_n y_1 y_2 \cdots y_m\,.$$

Every mapping $f : \mathcal{A} \longrightarrow M$ (M being a monoid) extends uniquely as a morphism of monoids $\bar{f} : \mathcal{A}^* \longrightarrow M$ by

$$\bar{f}(x_1 x_2 \cdots x_n) := f(x_1) f(x_2) \cdots f(x_n)\,.$$

A submonoid $M \subset \mathcal{A}^*$ need not be free, but always has a minimal set of generators $\mathcal{B}(M)$. In case M is free, $\mathcal{B}(M)$ is called the *base* or *code* of M and we have the identity between non-commutative power series

$$\sum_{w\in M} w = \frac{1}{1-C} \tag{29}$$

where $C := \sum_{w\in B(M)} w$ (see [23]). The expression $1/(1-C) = \sum_{n\geqslant 0} C^n$ will be denoted C^* in what follows.

We consider the two morphisms $\pi : \mathcal{A}^* \longrightarrow \mathcal{Z}$; $\mu : \mathcal{A}^* \longrightarrow End^{gr}(V)$ defined by

$$\pi(u) = -1 \qquad \pi(v) = 1 \qquad \mu(u) = f \qquad \mu(v) = g\,.$$

The extensions of π and μ from $\mathcal{A}$ to $\mathcal{A}^*$ are defined according to the laws of the monoids involved. Thus, the extension of π is additive, i.e. $\pi(w_1w_2) = \pi(w_1) + \pi(w_2)$, whereas the extension of μ is multiplicative, i.e. $\mu(w_1w_2) = \mu(w_1)\mu(w_2)$. Hence, $\pi(w)$ is equal to the difference between the number of appearances of the letter v and those of the letter u in the word w, and $\mu(w)$ is a product of the operators f and g ordered according to the ordering of u and v in w.

One has, for $w \in \mathcal{A}^*$

$$\mu(w) \in End^{\pi(w)}(V) \tag{30}$$

so that, if $\pi(w) = k$, we have $\mu(w)\cdot e_n \in V_{n+k}$. As previously, $\omega^{(i)}_{n\to n+k}$ is defined by

$$\left(\sum_{\substack{\pi(w)=k\\ |w|=i}} \mu(w)\right)\cdot e_n = \omega^{(i)}_{n\to n+k} e_{n+k} \tag{31}$$

and we consider the generating series

$$T_{n\to n+k} = \sum_{i=0}^{\infty} t^i \omega^{(i)}_{n\to n+k}\,.$$

We claim that $T_{n\to n+k}$ can be developed as a product of continued fractions, this property being inherited from the submonoid $\pi^{-1}(0)$ that consists of words in which v and u appear an equal number of times. Every word in $\pi^{-1}(0)$ can be written as a unique product of words belonging to the base D. A word belongs to D iff it cannot be partitioned into products of factors that contain an equal number of u and v factors.

Let us recall the following basic facts.

Proposition 5.1. (i) $\pi^{-1}(0)$ is a free submonoid of $\mathcal{A}^*$ with base

$$D = \{w|\pi(w) = 0,\ \ \forall w_1, w_2 :\ w = w_1w_2, w_1 \neq 1, w_2 \neq 1 \Longrightarrow \pi(w_1) \neq 0\} \tag{32}$$

(ii) If we set

$$D_+ = \{w|\pi(w) = 0,\ \ \forall w_1, w_2 :\ w = w_1w_2, w_1 \neq 1, w_2 \neq 1 \Longrightarrow \pi(w_1) > 0\}$$

$$D_- = \{w|\pi(w) = 0,\ \ \forall w_1, w_2 :\ w = w_1w_2, w_1 \neq 1, w_2 \neq 1 \Longrightarrow \pi(w_1) < 0\}$$

then $D = D_+ \cup D_-$ and

$$D_+ = v\left(\frac{1}{1-D_+}\right)u \qquad D_- = u\left(\frac{1}{1-D_-}\right)v\,. \tag{33}$$

Proof. These facts are well known. The submonoid $\pi^{-1}(0)$ being the preimage of unit by a morphism, is free (see, for example, [24], ch IV.5). □

Comment. It may be useful to note that by representing v, u as arrows directed to the right and up or down (say, $v \to (1, 1)$ and $u \to (1, -1)$), the words in D_+ are represented in terms of sequences of arrows, joined consecutively tail to head, that are always above some horizontal baseline at which they begin and terminate. Similarly, D_- consists of words that are represented by sequences of arrows that are always below that same baseline.

It follows from proposition 4.1 that $\pi^{-1}(0)$, as a series, is a double continued fraction. More generally, let $(\mathcal{A}^*)_k := \pi^{-1}(k) = \{w|\pi(w) = k\}$, and define the series F_n^+; F_n^- by

$$F_n^+ e_n := \sum_{w\in D_+^*} t^{|w|}\mu(w)\cdot e_n$$

$$F_n^- e_n := \sum_{w\in D_-^*} t^{|w|}\mu(w)\cdot e_n$$

$$F_n e_n := \sum_{w\in D^*} t^{|w|}\mu(w)\cdot e_n$$

then we have the following theorem.

Theorem 5.2. (i) If $k \geqslant 0$, one has

$$(\mathcal{A}^*)_k = (D_-^* v)^k D^* = D^*(vD_+^*)^k . \tag{34}$$

If $k < 0$

$$(\mathcal{A}^*)_k = (D_+^* u)^{-k} D^* = D^*(uD_-^*)^{-k} . \tag{35}$$

(ii) $F_n^{\pm}$ admits the following continued fraction representations

$$F_n^+ = \cfrac{1}{1-\cfrac{t^2\alpha_{n+1}\beta_{n+1}}{1-\cfrac{t^2\alpha_{n+2}\beta_{n+2}}{1-\cfrac{t^2\alpha_{n+3}\beta_{n+3}}{1-\cdots}}}} = \frac{1}{1-E_n^+}$$

$$F_n^- = \cfrac{1}{1-\cfrac{t^2\alpha_{n}\beta_{n}}{1-\cfrac{t^2\alpha_{n-1}\beta_{n-1}}{1-\cfrac{t^2\alpha_{n-2}\beta_{n-2}}{1-\cdots}}}} = \frac{1}{1-E_n^-} .$$

(iii) For F_n we get the form of a double continued fraction representation

$$F_n = \frac{1}{1-E_n^+-E_n^-} .$$

(iv) If $k \geqslant 0$, we have

$$T_{n\to n+k} = t^k F_{n+k} \prod_{i=0}^{k-1} F_{n+i}^- = t^k F_n \prod_{i=1}^{k} F_{n+i}^+ . \tag{36}$$

If $k \leqslant 0$

$$T_{n\to n+k} = t^{-k} F_{n+k} \prod_{i=0}^{-k-1} F_{n-i}^+ = t^{-k} F_n \prod_{i=1}^{-k} F_{n-i}^- . \tag{37}$$

Notice that F_n^- is a finite continued fraction.

As a particular case we get the q-Weyl algebra.

Corollary 5.3. For the q-Weyl algebra ($\alpha_n = \beta_n = ([n]_q)^{\frac{1}{2}}$) one has

$$F_n = \cfrac{1}{1 - \cfrac{t^2[n+1]_q}{1 - \cfrac{t^2[n+2]_q}{1 - \cfrac{t^2[n+3]_q}{1 - \cdots}}} - \cfrac{t^2[n]_q}{1 - \cfrac{t^2[n-1]_q}{1 - \cfrac{t^2[n-2]_q}{1 - \cdots}}}}.$$

In particular, for $n = 0$ and $q = 1$ (the classical Weyl algebra) we get

$$F_0(1) = \cfrac{1}{1 - \cfrac{t^2}{1 - \cfrac{2t^2}{1 - \cfrac{3t^2}{1 - \cdots}}}} = 1 + \sum_{k=1}^{\infty} (2k-1)!!\, t^{2k}$$

i.e. $\omega^{(2k)}_{0\to 0} = (2k-1)!!$.

An example of computation

Let us first compute the coefficient of (28), ($\omega^{(k)}_{n\to n+k-2}$), for the case of the classical Weyl algebra. Here

$$\omega^{(k)}_{n\to n+k-2} = (T_{n\to n+k-2}, t^k)$$

where (T, t^n) stands for 'the coefficient of t^n in the series T' (see [23]). Then

$$\omega^{(k)}_{n\to n+k-2} = \left(t^{k-2} F_n \prod_{i=1}^{k-2} F^+_{n+i}, t^k\right) = \left(F_n \prod_{i=1}^{k-2} F^+_{n+i}, t^2\right)$$

so one has just to compute $\bmod t^3$. But

$$F_n \prod_{i=1}^{k-2} F^+_{n+i} \equiv \left(1 + (n+n+1)t^2\right) \prod_{i=1}^{k-2} \left(1 + (n+i+1)t^2\right) \mod t^3$$

then, we obtain

$$\omega^{(k)}_{n\to n+k-2} = n + n + 1 + \sum_{i=1}^{k-2} (n+i+1) = kn + \frac{k(k-1)}{2}$$

which is equation (23). Similarly

$$\omega^{(k)}_{n\to n-k+2} = (T_{n\to n-k+2}, t^k)$$

and, again, one has just to compute $\bmod t^3$. Here, we use the formula

$$T_{n\to n-k+2} = t^{k-2} F_n \prod_{i=1}^{k-2} F^-_{n-i}$$

However

$$F_n \prod_{i=1}^{k-2} F^-_{n-i} \equiv \left(1 + (n+n+1)t^2\right) \prod_{i=1}^{k-2} \left(1 + (n-i)t^2\right) \mod t^3$$

hence

$$\omega^{(k)}_{n\to n-k+2} = \sum_{i=0}^{k-1} (n+1-i) = k\left(n - \frac{k-3}{2}\right)$$

which is equation (24). Equations (25)–(27) can also be recovered with this technique. As a further illustration we show that

$$F_n \equiv \cfrac{1}{1 - \cfrac{t^2[n+1]}{1-t^2[n+2]} - \cfrac{t^2[n]}{1-t^2[n-1]}}$$

$$\equiv 1 + (2n+1)t^2 + ((-n-1)(-n-2) - n(-n+1) + (-2n-1))t^4 \mod t^5 .$$

The coefficient of t^4 is $6n^2 + 6n + 3$, in agreement with equation (25).

6. Conclusions

In the present article the boson and q-boson ordering transformations are investigated as an exciting common ground of physics and combinatorics. The richness and variety of paths that are at our disposal in this context is displayed, with an emphasis on an appreciation of the interrelations between various transformations and between various techniques. The transformation from symmetric to normal ordering, which, for conventional boson operators, can most simply be effected using a special case of the CBH formula, stands out as a source of distinct difficulties when considered with respect to q-boson operators. To circumvent the lack of a suitable q-analogue of the CBH formula, two alternative procedures are presented, based on a recurrence relation and on a double continued fraction, respectively. These procedures enrich the repertoire of techniques available in this field.

It is appropriate to point out, in conclusion, that the q-deformed boson symmetric ordering is a crucial element in the development of a q-deformed Wigner representation. In fact, the structure of the results obtained in sections 4 and 5 should be carefully examined in order to uncover hints of further simplification that may possibly be achieved by redefining the notion of symmetric ordering for q-bosons. The most likely and natural revision should allow for q-weighting of the various terms in the symmetrically ordered boson operator. We hope to return to these considerations in the future.

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